Generalized Covering Designs

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This structure is a generalization of t-designs, where any t-subset is contained in exactly λ blocks.

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2 3 5
3 4 6
1 4 5
2 5 6
1 3 6
1 2 4

Definitions and Notation

Let $\mathbf{x} = (x_1, x_2, \ldots, x_m)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_m)$ both be m-tuples of integers. We say that $\mathbf{x} \leq \mathbf{y}$ if $x_i \leq y_i$ for each $i \in \{1, 2, ..m\}$.

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Let v, k, t, and λ be positive integers such that $v \geq k \geq t$. Let $\mathbf{v} = (v_1, v_2, \ldots, v_m)$ be an m-tuple of positive integers with sum v, and let $\mathbf{k} = (k_1, k_2, \ldots, k_m)$ be an m-tuple of positive integers with sum k, such that $\mathbf{k} \leq \mathbf{v}$.

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Let $\mathbf{X} = (X_1, X_2, \ldots, X_m)$ be pairwise disjoint sets, where $|X_i| = v_i$, and let $\mathbf{T}$ be an m-tuple of disjoint sets, each with sum t.

Generalized Covering Designs

A generalized covering design, $GC_\lambda(\mathbf{v}, \mathbf{k}, t)$ is a collection of elements of $\binom{\mathbf{X}}{\mathbf{k}}$, called blocks, such that each admissible $\mathbf{T}$ is contained in at least λ blocks.

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The covering number, $C(\mathbf{v}, \mathbf{k}, t)$, denotes the smallest number of blocks of a generalized covering design $GC(\mathbf{v}, \mathbf{k}, t)$.

Generalized Covering Designs

Let $\mathbf{v} = (3, 4)$, and $\mathbf{k} = (2, 3)$. Then, $\mathbf{X} = (\{1, 2, 3\}, \{1, 2, 3, 4\})$ and the admissible vectors for $\mathbf{t}$ are (2,0), (0,2), or (1,1). The following is a GC($\mathbf{v}, \mathbf{k}, 2$):

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$$
\begin{array}{ll}
(\{12\}, & \{123\}) \\
(\{13\}, & \{124\}) \\
(\{23\}, & \{234\})
\end{array}
$$

Point Deletion and Block Expansion

Suppose we have vectors $\mathbf{v}$ and $\mathbf{k}$ where $\mathbf{k} \leq \mathbf{v}$ and each $k_i \geq 2$. Let C be an optimal $(v_{max}, k_{min}, 2)$-covering design. Bailey et al. describe an algorithm to construct a $GC(\mathbf{v}, \mathbf{k}, 2)$ as follows:

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4. replace each placeholder greedily, ensuring that no symbol is repeated within any block;

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If there exists a $(v_i, k_i) = (v_{max}, k_{min})$ and C is optimal, then the resulting design is optimal.

Example

Suppose $\mathbf{v} = (6, 5)$ and $\mathbf{k} = (3, 4)$. Then $v_{max} = 6$, and $k_{min} = 3$.

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$$\begin{array}{ll} (\{235\}, & \{1235\}) \\ (\{346\}, & \{1234\}) \\ (\{145\}, & \{1245\}) \\ (\{256\}, & \{1235\}) \\ (\{136\}, & \{1234\}) \\ (\{124\}, & \{1234\}) \end{array}$$

Block Recursive Construction

Suppose D_1 is a generalized covering design $GC(\mathbf{v}, \mathbf{k}, t)$ with b blocks, and D_2 is a generalized covering design $GC(\mathbf{w}, \mathbf{l}, t)$ with c blocks, where $b \geq c$.

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In *Generalized Covering Designs and Clique Coverings*, Bailey et al. describe an algorithm that produces a $GC(\text{cat}(\mathbf{v}, \mathbf{w}),\text{cat}(\mathbf{k}, \mathbf{l}), t)$ with bc blocks formed by the concatenation of the blocks of D_1 and D_2.

Modified Block Recursive Construction

They also describe a modified block recursive construction, which produces a $GC(\text{cat}(\mathbf{v},\mathbf{w}),\text{cat}(\mathbf{k},\mathbf{l}),2)$ with at most

$$\max\{C(\mathbf{v},\mathbf{k},2), C(\mathbf{w},\mathbf{l},2)\} + \left(\max\left\lceil \frac{v_i}{k_i} \right\rceil\right)\left(\max\left\lceil \frac{w_i}{l_i} \right\rceil\right)$$

blocks.

MacNeish-Type Construction

Let $\mathbf{v} = (v_1, v_2, \ldots, v_m)$ and $\mathbf{w} = (w_1, w_2, \ldots, w_m)$. The Hadamard product of $\mathbf{v}$ and $\mathbf{w}$ is the vector

$$\mathbf{v} \circ \mathbf{w} = (v_1 w_1, v_2 w_2, \ldots, v_m w_m).$$

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$$\mathbf{v} \circ \mathbf{w} = (v_1 w_1, v_2 w_2, \ldots, v_m w_m).$$

Let $B = \{\mathbf{B}_1, \ldots, \mathbf{B}_b\}$ be the b blocks of a $GC(\mathbf{v}, \mathbf{k}, 2)$, and $C = \{\mathbf{C}_1, \ldots, \mathbf{C}_c\}$ be the c blocks of a $GC(\mathbf{w}, \mathbf{l}, 2)$. The MacNeish-type construction produces a $GC(\mathbf{v} \circ \mathbf{w}, \mathbf{k} \circ \mathbf{l}, 2)$ with bc blocks $D = \{\mathbf{B}_i \circ_v \mathbf{C}_j | i = 1, \ldots, b, j = 1, \ldots, c\}$.

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Using the MacNeish-type construction, we could take the product of the blocks of a $GC((4,4),(2,2),2)$ and a $GC((3,2),(2,2),2)$:

$$\begin{array}{ll} (\{12\}, & \{13\}) \\ (\{13\}, & \{12\}) \\ (\{14\}, & \{14\}) \\ (\{23\}, & \{23\}) \\ (\{24\}, & \{24\}) \\ (\{34\}, & \{34\}) \end{array} \qquad \begin{array}{ll} (\{12\}, & \{12\}) \\ (\{23\}, & \{12\}) \\ (\{13\}, & \{12\}) \end{array}$$

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$1+(2-1)(4), 3+(2-1)(4), 1+(3-1)(4), 3+(2-1)(4)$

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$1+(1-1)(4), 2+(1-1)(4), 1+(2-1)(4), 2+(2-1)(4)$

MacNeish-Type Construction

$(\{13\}, \{12\}) \circ_v (\{23\}, \{12\})$

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$1 + (1-1)(4), 2 + (1-1)(4), 1 + (2-1)(4), 2 + (2-1)(4)$

$(\{5, 7, 9, 11\}, \{1, 2, 5, 6\})$

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$(\{13\}, \{12\}) \circ_v (\{23\}, \{12\})$

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$(\{5, 7, 9, 11\}, \{1, 2, 5, 6\})$

This would give a $GC((12, 8), (4, 4), 2)$ with $6 \times 3 = 18$ blocks.

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1. there exists a $(v_i, k_i) = (v_{max}, k_{min})$, then the point deletion and block expansion algorithm is optimal;
2. any entry of **v** or **k** is prime, we cannot use the MacNeish-type construction.

Construction Algorithms

Let $v_1 \geq v_2$, and $k_1 \leq k_2$. Suppose we fix v_1, k_1, and k_2. Let $v_2 = x$ be increasing. Let $P(x)$, $B(x)$, and $M(x)$ be the number of blocks given by the point deletion and block expansion, block recursive, and MacNeish-type constructions respectively.

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We know that $B(x) = \max\{C(v_1, k_1, 2), C(x, k_2, 2)\} + \left\lceil \frac{v_1}{k_1} \right\rceil \times \left\lceil \frac{v_2}{k_2} \right\rceil$ and $P(x) = C(v_1, k_2, 2)$.

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We know that $B(x) = \max\{C(v_1, k_1, 2), C(x, k_2, 2)\} + \left\lceil \frac{v_1}{k_1} \right\rceil \times \left\lceil \frac{v_2}{k_2} \right\rceil$ and $P(x) = C(v_1, k_2, 2)$.

$B(x)$ is increasing and $P(x)$ is constant.

Construction Algorithms

Suppose we want a $GC((12, v_2), (6, 4), 2)$. Then, we get the following values for $B(v_2)$, $P(v_2)$, and $M(v_2)$.

v_2	$P(v_2)$	$B(v_2)$	$M(v_2)$
4	12	5	3
5	12	7	
6	12	7	3
7	12	9	
8	12	10	6
9	12	14	9
10	12	15	10
11	12	17	
12	12	18	15

Goals

When v_{max} and k_{min} are not in the same part, the point deletion and block expansion algorithm is not always best. We would like to identify a set of parameters for which the block recursive construction or the MacNeish-type construction is always optimal. We would also like to compose programs to produce a generalized covering design from each construction.