Hawking Radiation

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Outline

- Origin of Hawking radiation,

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- The particle perspective of Hawking radiation,

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- The temperature of the black holes.

Introduction

The first publication of the Hawking radiation is the article *Black hole explosions?* on *Nature* Vol. 248, 1974.

Stephen Hawking discovered that black holes have certain temperature, which results in certain mass loss or radiation from black holes.

Such discovery was very controversial. Even Hawking himself had doubts about the physical cause of the radiation. He thought of the possibility that the radiation come from artifact of the mathematical structures.

Black Hole and Thermaldynamics

The four laws of the black hole mechanics and the four laws of thermaldynamics are connected in forms.

The first law of black hole mechanics is

$$\delta M = \frac{1}{8\pi}\kappa\delta A + \Omega\delta J + \Phi\delta Q, \tag{1}$$

where δM is the change in mass, κ is surface gravity, δA is change in area of the event horizon, Ω is angular velocity, δJ is change in angular momentum, Φ is electrostatic potential and δQ is change in charge. The expression is similar to the first law of thermal dynamics

$$\delta U = Q - W, \tag{2}$$

Second Law

Black hole mechanics: the event horizon area A cannot be reduced.

Thermodynamics : the entropy of a closed system cannot be reduced.

Third Law

Black hole mechanics: the surface gravity κ cannot be reduced to zero by finite amount of operations.

Thermodynamics : Nernst form is that the temperature of a system cannot be reduced to absolute zero in finite amount of operations.

Forth (or Zeroth) Law

Black hole mechanics: the surface gravity κ of a stationary black hole is constant on the entire surface of the event horizon.

Thermodynamics : if two thermodynamic systems are each in thermal equilibrium with a third, then they are in thermal equilibrium with each other.

Problems with Hawking radiation

- 1. The generalized second law issue: GSL stated that the total entropy of a black hole and its surroundings cannot decrease. This law can be violated by an object with minimal energy near a black hole, so that the increase of black hole entropy would not make up for its own entropy lose.

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- 3. Unitary issue: When a emitted photon fall into the black hole, which will rise a density matrix description of the radiation. This is a non-unitary description of the evolution, which is not a typical quantum mechanic process.

"Black hole explosion?"

Consider a massless Hermitean scalar field *phi* in an asymptotially flat space time containing a star which is collapsing into a black hole.

$$\phi = \sum_i (f_i a_i + \bar{f}_i a_i^+), \tag{3}$$

where f_i are a complete orthornalmal family of wave equation solutions and a_i and a_i^+ are the annihilation and creation operator. ϕ can also be written as

$$\phi = \sum_i (p_i b_i + \bar{p}_i b_i^+ + q_i c_i + \bar{q}_i c_i), \tag{4}$$

where p_i are the solutions of the wave equations which have positive frequency and are zero on the event horizon and outgoing and q_i are the solutions which are zero on the future null infinity I^+.

"Black hole explosion?"

p_i and q_i can be further expressed in terms of f_i and $\bar{f}_i$, such as

$$p_i = \sum_j (\alpha_{ij} f_i + \beta_{ij} \bar{f}_i)$$

By comparing Eqn 3 and 4, one can write

$$b_i = \sum_j (\bar{\alpha}_{ij} a_j - \bar{\beta}_{ij} a_j^+) \quad (5)$$

When there is no incoming particles in the expectation value, the outgoing state of $b_i^+ b_i$ is

$$< 0_- | b_i^+ b_i | 0_- >= \sum_j |\beta_{ij}|^2$$

"Black hole explosion?"

The outgoing solution p_{lmw} can be express as (with fixed lm)

$$p_\omega = \int (\alpha_{\omega\omega'} f_{\omega'} + \beta_{\omega\omega'} \bar{f}_{\omega'}) d\omega' \tag{6}$$

To solve $\alpha_{\omega\omega'}$ and $\beta_{\omega\omega'}$, imagine a wave with positive frequency ω on I^+ and propagating backwards via spacetime in solitude through event horizon and some of it will be scattered by the event horizon and escape to I^- with the same frequency ω.

"Black hole explosion?"

The other part of the wave will propagate backwards into the collapsing star and eventually out onto I^- with significant blue shift with the form

$$C\omega^{-1/2}\exp(-i\omega\kappa^2\log(v_0 - v) + i\omega v) \quad \text{for} \quad v < v_0$$

and

$$0 \quad \text{for} \quad v \geq v_0,$$

where v_0 is the last advanced time for a particle leave I^-, return to the origin and move to I^+.

“Black hole explosion?”

Eventually, one can solve $\alpha_{\omega\omega'}$ and $\beta_{\omega\omega'}$ and find

$$|\alpha_{\omega\omega'}| = \exp(\pi\omega/\kappa)|\beta_{\omega\omega'}|, \tag{7}$$

which indicates that the wave pocket mode would emit different amount of particles than it would absorb during a similar event on the black hole from I^-, with the ratio of

$$\frac{1}{\exp(2\pi\omega/\kappa) - 1}.$$

This means the particles are emitted from the black hole in total.

Cross section and Decay rate

The total decay rate

$$\Gamma_{tot} = \sum_{i=1}^{n} \Gamma_i,$$

The expression for the decay rate Γ_i is

$$\Gamma_i = \int |\mathscr{M}|^2 (2\pi)^4 \delta^4(p_1 - p_2 - \cdots - p_n) \prod_{j=2}^{n} \frac{d^3 p_j}{(2\pi)^3 2E_j}, \tag{8}$$

which is known as the Fermi's golden rule. It consists of 3 main components-amplitude, phase space factor and the delta function.

Cross section Example

the amplitude expression of the electron-muon scattering can be constructed as

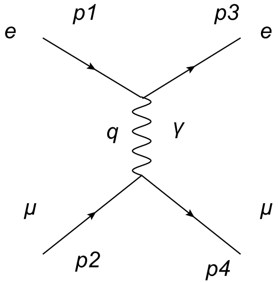

Figure: Feynman diagram for electron-muon scattering.

$$\begin{aligned}\mathscr{M} =&(2\pi)^4 \int [\bar{u}_e(p_3, s_3) i g_e \gamma^\mu u_e(p_1, s_1)] \frac{-i g_{\mu\nu}}{q^2} [\bar{u}_{muon}(p_4, s_4) i g_e \gamma^\nu u_{muon}(p_2, s_2)] \\ &\times \delta^4(p_1 - p_3 - q)\delta^4(p_2 - p_4 + q) d^4 q\end{aligned} \tag{9}$$

Particle perspective of Hawking radiation

The Hawking radiation can be considered as a black hole decay. The black hole loses a massless quantum, such as photon, and become a new black hole.

$$(\text{black hole})_i \rightarrow (\text{black hole})_f + \gamma \tag{10}$$

Particle perspective of Hawking radiation

The estimated expression for the decay rate of Hawking radiation is

$$\hbar\Gamma(b_i \to b_f + \gamma) = \epsilon \left(\frac{\hbar\omega}{m_P}\right)^2 \left(\frac{A_H}{\ell_p^2}\right)(\hbar\omega)\left(2\frac{2J_f+1}{2J_i+1}\right)\frac{N_f}{N_i}, \tag{11}$$

with the units of $c = 1$ and $G = \frac{\ell_p}{m_p} = \frac{\ell_p^2}{\hbar} = \frac{\hbar}{m_p^2}$.

Particle perspective of Hawking radiation

$$\hbar\Gamma(b_i \to b_f + \gamma) = \epsilon \left(\frac{\hbar\omega}{m_P}\right)^2 \left(\frac{A_H}{\ell_p^2}\right)(\hbar\omega)\left(2\frac{2J_f+1}{2J_i+1}\right)\frac{N_f}{N_i},$$

It consists of 6 terms and one assumption.

- ϵ is a free scalar parameter for normalization.
- $\left(\frac{\hbar\omega}{m_P}\right)$ is the coupling constant of the decay process. To be more precise, it is the coupling between matter and gravity,
- $\left(\frac{A_H}{\ell_p^2}\right)$ is the ratio between the area of the event horizon and the areal of Planck length square. It provides a scale for how many individual decay can occur all over the event horizon, since each unit area of ℓ_P^2 can emit radiation separately. It works as the sum of all Γ_i

Particle perspective of Hawking radiation

$$\hbar\Gamma(b_i \to b_f + \gamma) = \epsilon \left(\frac{\hbar\omega}{m_P}\right)^2 \left(\frac{A_H}{\ell_p^2}\right)(\hbar\omega)\left(2\frac{2J_f+1}{2J_i+1}\right)\frac{N_f}{N_i},$$

- $\hbar\omega$ is the energy of the radiation, which is proportional to the phase space component of a typical decay process.
- $\left(2\frac{2J_f+1}{2J_i+1}\right)$ is the spin factors. This term comes from the sum over the squares of Clebsch-Gordon coefficients. Clebsch-Gordon coefficients are related to the angular momentum components of the amplitude. i.e. the s_i terms in Eqn 1. For the most simple version of the decay, one can set $J_i = J_f = 0$.

Particle perspective of Hawking radiation

$$\hbar\Gamma(b_i \to b_f + \gamma) = \epsilon \left(\frac{\hbar\omega}{m_P}\right)^2 \left(\frac{A_H}{\ell_p^2}\right)(\hbar\omega)\left(2\frac{2J_f+1}{2J_i+1}\right)\frac{N_f}{N_i},$$

- The last term $\frac{N_f}{N_i}$ is the statistical factors. It averages over the initial stats N_i and sums over the final states N_f.
- The one assumption it has was the conservation law holds, which serves the same purpose as the delta function terms.

Particle perspective of Hawking radiation

$$\frac{N_f}{N_i} = \exp(S_f - S_i) \approx \exp\left(\frac{\partial S}{\partial M}[M_f - M_i]\right) \approx \exp\left(-\frac{\partial S}{\partial M}\hbar\omega\right) = \exp(-\hbar\omega/T) \tag{12}$$

Now, we can rewrite Eqn 11 as

$$\Gamma(b_i \to b_f + \gamma) = \epsilon\left(\frac{\hbar\omega}{m_P}\right)^2\left(\frac{A_H}{\ell_p^2}\right)\omega\left(2\frac{2J_f+1}{2J_i+1}\right)\exp(-\hbar\omega/T), \tag{13}$$

with the $d\omega$ increment at ω

$$d\Gamma(b_i \to b_f + \gamma) = \epsilon\left(\frac{\hbar\omega}{m_P}\right)^2\left(\frac{A_H}{\ell_p^2}\right)\left(2\frac{2J_f+1}{2J_i+1}\right)\exp(-\hbar\omega/T)d\omega$$

Particle perspective of Hawking radiation

Therefore the power from the decay/radiation is

$$dP(\omega) = \hbar\omega d\Gamma(b_i \to b_f + \gamma) = 2\epsilon\left(\hbar\omega\right)^3 A_H \left(\frac{2J_f + 1}{2J_i + 1}\right)\exp(-\hbar\omega/T)d\omega$$

Let $J_f = J_i = 0$ and compare it to the original paper by Hawking, we have $\epsilon = \frac{1}{3\pi}$ and $T = \kappa$.

Conclusion

It is an interesting combination of both quantum mechanics and general relativity. However, Hawking radiation only produce a limited effect on black holes to be observed astronomically. On the other hand, it is impossible to simulate black hole effects experimentally in the near future.

With more developments on the gravitational detection field, the chance of discovering Hawking radiation related data could be increasing. The gravitational coupling constant obtained from other sources can provide more precise model for Hawking radiation. In return, a better Harking radiation model can help introducing additional massless quanta, such as graviton to the Standard model.

Questions?